

Characterizing closed curves on Riemann surfaces via homology groups of coverings

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Abstract

Let $S_{g,n}$, for $2g - 2 + n > 0$, be a closed oriented Riemann surface of genus g from which n points have been removed. The purpose of this paper is to show that closed curves on $S_{g,n}$ are characterized by the submodules they determine in the homology groups of finite unramified coverings of $S_{g,n}$.

More precisely, for a given finite unramified covering $\pi: S \rightarrow S_{g,n}$, let us denote by \bar{S} the closed Riemann surface obtained filling in the punctures of S . Then, for a given closed curve γ on $S_{g,n}$, the cycles supported on the irreducible components of $\pi^{-1}(\gamma)$ (see Definition 2.1) span a submodule V_γ of the homology group $H_1(\bar{S}, \mathbb{Z})$.

The main result of the paper is a characterization of simple closed curves on $S_{g,n}$. A non-power closed curve γ on $S_{g,n}$ is homotopic to a simple closed curve if and only if, for a fixed prime p , every finite unramified p -covering $\pi: S \rightarrow S_{g,n}$ is such that the associated submodule V_γ of $H_1(\bar{S}, \mathbb{Z})$ is isotropic for the standard intersection pairing on the closed Riemann surface \bar{S} .

We then prove that, if γ and γ' are two non homotopic simple closed curves on $S_{g,n}$, then there is a finite unramified p -covering $\pi: S \rightarrow S_{g,n}$ such that $V_\gamma \neq V_{\gamma'}$ in the homology group $H_1(\bar{S}, \mathbb{Z})$. As an application, we give a geometric argument to prove that oriented surface groups are conjugacy p -separable (a combinatorial proof of this fact was recently given by Paris [9]).

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1 Introduction

In the papers [8] and [12], Stallings and Jaco established the equivalence between the Poincaré conjecture, now a celebrated theorem by Perelman, and the following group-theoretical statement:

- ★ *Let S_g be a closed oriented surface of genus $g \geq 2$. Let F_g be a free group of rank g . Let $\eta: \pi_1(S_g, s_0) \rightarrow F_g \times F_g$ be an epimorphism. Then, there is a non-trivial element in the kernel of η which may be represented by a simple closed curve in S_g .*

Of course, it is still a very interesting problem to provide a group-theoretic proof of the above statement. The first step in this direction is to give an algebraic characterization of simple closed curves on the closed Riemann surface S_g .

A program in this sense was formulated by Turaev [13]. Progresses in this direction have been recently accomplished by Chas, Krongold and Sullivan (see [2], [3], [4], [5]), who characterize simple closed curves on a Riemann surface with boundary in terms of the Goldman Lie algebra introduced in [7].

In Section 2, we give an elementary criterion to characterize simple closed curves on any hyperbolic Riemann surface in terms of the intersection pairing on the closures of finite unramified p -coverings of the given Riemann surface, for a fixed prime p . The proof is based on the elementary hyperbolic geometry of the p -adic solenoid and some profinite group theory.

A consequence of this result is that, given an element $\gamma \in \pi_1(S_g, s_0)$, whether the free homotopy class of γ contains or not an embedded representative can be determined in terms of the group structure of the fundamental group $\pi_1(S_g, s_0)$, thus providing a clue that a purely group-theoretic proof of statement \star is, in principle, possible.

In Section 4, we apply the above characterization of simple closed curves to show that it is possible to distinguish homotopy classes of closed curves on a hyperbolic Riemann surface in terms of the first homology groups of its finite unramified p -coverings. An easy consequence is conjugacy p -separability of surface groups.

2 Characterization of simple closed curves

Let S_g be a compact oriented Riemann surface without boundary of genus g and let $S_{g,n} := S_g \setminus \{P_1, \dots, P_n\}$ be the same surface from which n distinct points have been removed. We assume that $\chi(S_{g,n}) = 2 - 2g - n < 0$. Even if most of the stated results hold also in the non-oriented case, for simplicity, we only consider the oriented case.

A closed curve on $S_{g,n}$ is a continuous map from the circle S^1 to the surface $S_{g,n}$. For a fixed base point, let us denote by $\Pi_{g,n}$ the fundamental group of $S_{g,n}$. Then, it is well known that the set of homotopy classes of closed curves on $S_{g,n}$ identifies with the quotient of the sets $\{\{\alpha, \alpha^{-1}\} \mid \alpha \in \Pi_{g,n}\}$ and $\{\langle \alpha \rangle \mid \alpha \in \Pi_{g,n}\}$ by the action induced by inner automorphisms of $\Pi_{g,n}$. It is instead a deeper result, known as Magnus property (see [1]), that the set of closed curves on $S_{g,n}$ is also in natural bijective correspondence with the set $\{\langle \alpha \rangle_N \mid \alpha \in \Pi_{g,n}\}$, where, by $\langle x \rangle_N$, we denote the normal subgroup normally generated by an element x of $\Pi_{g,n}$.

Definition 2.1. For a normal finite index subgroup K of $\Pi_{g,n}$, let $\pi_K: S_K \rightarrow S_{g,n}$ be the associated covering, G_K its deck transformation group and \bar{S}_K the compact Riemann surface obtained filling in the punctures of S_K . Given a closed curve γ on $S_{g,n}$, let $\tilde{\gamma}$ be an element of $\Pi_{g,n}$ whose free homotopy class contains the closed curve γ and let $k > 0$ be the smallest integer such that $\tilde{\gamma}^k \in K$. Let us fix a base point on the surface S_K , above the base point of $\Pi_{g,n}$, and let us identify its fundamental group with the subgroup K .

An *irreducible component* of the inverse image $\pi_K^{-1}(\gamma)$ of γ is a closed curve supported on $\pi_K^{-1}(\gamma)$ and contained in the free homotopy class of some $\Pi_{g,n}$ -conjugate of $\tilde{\gamma}^k$.

It is clear that the inverse image $\pi_K^{-1}(\gamma)$ of a closed curve γ is the union of its irreducible components and that these do not depend upon the choice of $\tilde{\gamma} \in \Pi_{g,n}$.

Let γ be a closed curve on $S_{g,n}$ and let $\tilde{\gamma} \in \Pi_{g,n}$ be an element of the fundamental group whose free homotopy class contains γ . Then, for an integer $s > 0$, a *power* γ^s of γ is a closed curve on $S_{g,n}$ contained in the free homotopy class of the element $\tilde{\gamma}^s \in \Pi_{g,n}$ (for $s = 1$, this just means that γ^1 is a closed curve homotopic to γ). A closed curve on the Riemann surface $S_{g,n}$ is *non-power* if there is a primitive element of $\Pi_{g,n}$ whose free homotopy class contains the given curve.

Let us observe that, with the notations of Definition 2.1, given a closed curve γ and a power γ^s with the same support of γ and such that s divides k , the inverse images $\pi_K^{-1}(\gamma)$ and $\pi_K^{-1}(\gamma^s)$ have the same irreducible components. More generally, if γ^s is a power of γ and m is the m.c.d. of s and k , then each irreducible component of $\pi_K^{-1}(\gamma^s)$ is an s/m -power of one of $\pi_K^{-1}(\gamma)$.

Definition 2.2. Let $H_1(\overline{S}_K)$ be the first homology group of \overline{S}_K with \mathbb{Z} -coefficients. Let then $V_{K,\gamma}$ be the G_K -invariant submodule of $H_1(\overline{S}_K)$ spanned by the cycles supported on the irreducible components of $\pi_K^{-1}(\gamma)$.

In particular, we see that the submodule $V_{K,\gamma}$ is contained in the image of $\langle \tilde{\gamma} \rangle_N \cap K$ inside the homology group $H_1(\overline{S}_K)$. However, in general, this is a strict inclusion.

A simple closed curve (briefly s.c.c.) on $S_{g,n}$ is an embedded circle $S^1 \hookrightarrow S_{g,n}$. An interesting problem is that of establishing when a given closed curve γ on $S_{g,n}$ has in its homotopy class an embedded representative. In this section, we will give a characterization of this property in terms of the homology of finite unramified coverings of $S_{g,n}$.

More generally, we will be able to determine in this way when the geometric intersection number $|\gamma \cap \gamma'|_G$ between two closed curves γ and γ' is zero.

Theorem 2.3. *Let γ and γ' be closed curves on a Riemann surface $S_{g,n}$. Then, $|\gamma \cap \gamma'|_G = 0$ if and only if, for a fixed prime p and a cofinal system of p -power index normal subgroups $\{K\}$ of $\Pi_{g,n}$, it holds $\langle x, y \rangle_K = 0$, for all $x \in V_{K,\gamma}$ and all $y \in V_{K,\gamma'}$, where $\langle -, - \rangle_K$ is the intersection pairing on the first integral homology group of the closed Riemann surface \overline{S}_K . The same statement holds for any cofinal system of finite index normal subgroups of $\Pi_{g,n}$.*

As a corollary of Theorem 2.3, we get the aforementioned characterization of s.c.c.'s:

Corollary 2.4. *The homotopy class of a non-power closed curve γ on $S_{g,n}$ contains a simple closed curve if and only if, for a fixed prime p and a cofinal system of p -power index normal subgroups $\{K\}$ of $\Pi_{g,n}$, the associated submodules $V_{K,\gamma}$ of $H_1(\overline{S}_K)$ are isotropic. The same statement holds for any cofinal system of finite index normal subgroups of $\Pi_{g,n}$.*

The proof of Theorem 2.3 can be divided in two lemmas.

Lemma 2.5. *Let γ be a non-power closed curve on the Riemann surface $S_{g,n}$ not bounding a disc with less than two punctures. Then, for any fixed prime p , there is a normal, unramified p -covering $\pi_L: S_L \rightarrow S_{g,n}$ such that each irreducible component of $\pi_L^{-1}(\gamma)$ is homotopic to a non-separating simple closed curve.*

Remark 2.6. The existence of a finite unramified covering with the properties stated in Lemma 2.5 follows more directly from the fact that finitely generated subgroups of a surface group $\Pi_{g,n}$ are geometric, i.e. they can be realized as fundamental groups of subsurfaces of finite coverings of $S_{g,n}$ (see [10] and [11]). It is possible that also the last part of the statement of Theorem 2.3 can be proved in the same way. In any case, for trivial reasons, a p -version of Scott's results does not hold.

Proof. Let us fix a non-singular metric on $S_{g,n}$ such that the punctures are removed cusps and let us assume that the curve γ is the geodesic representative in its homotopy class.

Let \mathbb{D} be the universal cover of $S_{g,n}$. It can be identified with the hyperbolic disc. Let $\tilde{\gamma}$ be a lift of γ in \mathbb{D} which is then also a geodesic and the axe of an element $\tilde{\gamma} \in \Pi_{g,n}$ whose free homotopy class contains γ .

Let $\Pi_{g,n}^{(p)}$ be the pro- p completion of the fundamental group $\Pi_{g,n}$. There is a natural injective homomorphism $\Pi_{g,n} \hookrightarrow \Pi_{g,n}^{(p)}$, by which, we identify $\Pi_{g,n}$ with a subgroup of $\Pi_{g,n}^{(p)}$.

The p -adic solenoid $\mathbb{S}_{g,n}^{(p)}$ is defined to be the inverse limit $\varprojlim_{K \triangleleft \Pi_{g,n}} S_K$, where K varies among all normal subgroups of $\Pi_{g,n}$ of index a power of p . This is the p -adic version of the hyperbolic n -punctured, genus g solenoid (for a survey on this topic, see Šarić [14]) and is naturally isomorphic to the quotient $\mathbb{D} \times \Pi_{g,n}^{(p)} / \Pi_{g,n}$, where an element $x \in \Pi_{g,n}$ acts on the product $\mathbb{D} \times \Pi_{g,n}^{(p)}$ by the formula $x \cdot (d, \beta) = (x \cdot d, x\beta)$.

The p -adic solenoid comes with a natural embedding $\mathbb{D} \hookrightarrow \mathbb{S}_{g,n}^{(p)}$ and, under the above isomorphism, the image of the hyperbolic plane in the solenoid identifies with the image of $\mathbb{D} \times \Pi_{g,n}$ in the quotient $\mathbb{D} \times \Pi_{g,n}^{(p)} / \Pi_{g,n}$.

The axe $\tilde{\gamma}$ identifies with the subspace $\tilde{\gamma} \times \langle \tilde{\gamma} \rangle / \langle \tilde{\gamma} \rangle$ of $\mathbb{D} \times \Pi_{g,n}^{(p)} / \Pi_{g,n}$. Let us show that its closure identifies with the quotient of the closed subspace $\tilde{\gamma} \times \overline{\langle \tilde{\gamma} \rangle}$ of $\mathbb{D} \times \Pi_{g,n}^{(p)}$ by the action of its stabilizer $\overline{\langle \tilde{\gamma} \rangle} \cap \Pi_{g,n}$, where we denote by $\overline{\langle \tilde{\gamma} \rangle} \cong \mathbb{Z}_p$ the closed subgroup of $\Pi_{g,n}^{(p)}$ topologically generated by $\tilde{\gamma}$. Since the curve γ is non-power, it holds $\overline{\langle \tilde{\gamma} \rangle} \cap \Pi_{g,n} = \langle \tilde{\gamma} \rangle$. There is then a natural continuous map:

$$\varphi_\gamma: \tilde{\gamma} \times \overline{\langle \tilde{\gamma} \rangle} / \langle \tilde{\gamma} \rangle \rightarrow \mathbb{D} \times \Pi_{g,n}^{(p)} / \Pi_{g,n} \equiv \mathbb{S}_{g,n}^{(p)}.$$

This map is injective since two points (d, β) and (d', β') of $\tilde{\gamma} \times \overline{\langle \tilde{\gamma} \rangle}$ are mapped to the same point of the p -adic solenoid if and only if there is an $x \in \Pi_{g,n}$ such that $(x \cdot d, x\beta) = (d', \beta')$ and this is possible only if $x \in \overline{\langle \tilde{\gamma} \rangle} \cap \Pi_{g,n} = \langle \tilde{\gamma} \rangle$.

Let us then observe that the quotient space $\tilde{\gamma} \times \overline{\langle \tilde{\gamma} \rangle} / \langle \tilde{\gamma} \rangle$ is isomorphic to the quotient $\mathbb{R} \times \mathbb{Z}_p / \mathbb{Z}$, where $z \in \mathbb{Z}$ acts on $\mathbb{R} \times \mathbb{Z}_p$ by the formula $z \cdot (r, s) = (z + r, z + s)$, and this is a compact space. In particular, the image of φ_γ is a closed subspace of the p -adic solenoid.

Since the image of the map φ_γ contains the axe $\tilde{\gamma}$ as a dense subset, it follows that it is actually the closure of the axe $\tilde{\gamma}$ in the p -adic solenoid $\mathbb{S}_{g,n}^{(p)}$.

For a given finite index normal subgroup K of $\Pi_{g,n}$, let us denote by γ_K the image of $\tilde{\gamma}$ in S_K . The inverse limit $\varprojlim_{K \triangleleft \Pi_{g,n}} \gamma_K$ is also naturally identified with the closure of $\tilde{\gamma}$ in the p -adic solenoid $\mathbb{S}_{g,n}^{(p)}$. Therefore, there is a series of isomorphisms:

$$\varprojlim_{K \triangleleft \Pi_{g,n}} \gamma_K \cong \tilde{\gamma} \times \overline{\langle \tilde{\gamma} \rangle} / \langle \tilde{\gamma} \rangle \cong \mathbb{R} \times \mathbb{Z}_p / \mathbb{Z}.$$

In particular, we see that the pro-curve $\varprojlim_{K \triangleleft \Pi_{g,n}} \gamma_K$ has no self-intersections, i.e. it holds $\varprojlim_{K \triangleleft \Pi_{g,n}} (\gamma_K \cap_s \gamma_K) = \emptyset$, where, for a geodesic δ on a Riemann surface, we let $\delta \cap_s \delta$ be its self-intersection points. Since, for all finite index normal subgroups K of $\Pi_{g,n}$, the set $\gamma_K \cap_s \gamma_K$ is finite and the inverse limit of a system of non-empty finite sets is non-empty, we conclude that there is a normal subgroup H of $\Pi_{g,n}$ of index a power of p such that it holds $\gamma_H \cap_s \gamma_H = \emptyset$.

It is easy to construct a regular unramified p -covering $S'_H \rightarrow S_H$ such that the inverse image of any s.c.c. on S_H is non-separating on S'_H . We let then L be a normal subgroup of $\Pi_{g,n}$ of index a power of p such that S_L dominates S'_H . \square

Lemma 2.7. *The statement of Theorem 2.3 holds for non-separating simple closed curves.*

Proof. Let us keep the conventions introduced in the proof of Lemma 2.5 and let us assume that both γ and γ' are geodesics for the given metric on $S_{g,n}$. If $\gamma = \gamma'$ or γ is disjoint from γ' , there is nothing to prove. Let us then suppose that $\gamma \neq \gamma'$ and $\gamma \cap \gamma' \neq \emptyset$.

There are lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ of γ and γ' , respectively, to \mathbb{D} such that $\tilde{\gamma} \neq \tilde{\gamma}'$ and $\tilde{\gamma} \cap \tilde{\gamma}' \neq \emptyset$. These are also geodesics in \mathbb{D} and are the axes of two elements $\tilde{\gamma}$ and $\tilde{\gamma}'$ of $\Pi_{g,n}$ whose free homotopy classes contain, respectively, γ and γ' . Then, the axes $\tilde{\gamma}$ and $\tilde{\gamma}'$ intersect in a single point P . Let us denote by \overline{P} the image of P in the surface $S_{g,n}$.

For a given finite index normal subgroup K of $\Pi_{g,n}$, let us denote by γ_K , γ'_K and P_K , respectively, the images of $\tilde{\gamma}$, $\tilde{\gamma}'$ and P in the surface S_K .

Let us identify as above $\hat{\gamma} := \varprojlim_{K \triangleleft \Pi_{g,n}} \gamma_K$ and $\hat{\gamma}' := \varprojlim_{K \triangleleft \Pi_{g,n}} \gamma'_K$ with the closures of $\tilde{\gamma}$ and $\tilde{\gamma}'$ in the p -adic hyperbolic solenoid $\mathbb{S}_{g,n}^{(p)}$. We claim that:

$$\varprojlim_{K \triangleleft \Pi_{g,n}} \gamma_K \bigcap \varprojlim_{K \triangleleft \Pi_{g,n}} \gamma'_K = \varprojlim_{K \triangleleft \Pi_{g,n}} (\gamma_K \cap \gamma'_K) = \{P\}. \quad (*)$$

In the proof of Lemma 2.5, we have seen that the pro-curves $\hat{\gamma}$ and $\hat{\gamma}'$ identify with the images of $\tilde{\gamma} \times \overline{\langle \tilde{\gamma} \rangle}$ and $\tilde{\gamma}' \times \overline{\langle \tilde{\gamma}' \rangle}$ in the quotient $\mathbb{D} \times \Pi_{g,n}^{(p)} / \Pi_{g,n}$.

For $s, t \in \mathbb{Z}_p$, the images of two leaves $\tilde{\gamma} \times \tilde{\gamma}^s$ and $\tilde{\gamma}' \times \tilde{\gamma}'^t$ in the quotient $\mathbb{D} \times \Pi_{g,n}^{(p)} / \Pi_{g,n}$ can possibly intersect only if there exists an $f \in \Pi_{g,n}$ such that $f \cdot \tilde{\gamma}'^t = \tilde{\gamma}^s$, i.e. only if:

$$\tilde{\gamma}^s \tilde{\gamma}'^{-t} \in \Pi_{g,n} \cap \overline{\langle \tilde{\gamma} \rangle} \cdot \overline{\langle \tilde{\gamma}' \rangle}.$$

Let us observe that, since $\tilde{\gamma}' \neq \tilde{\gamma}^{\pm 1}$, it holds $\overline{\langle \tilde{\gamma} \rangle} \cap \overline{\langle \tilde{\gamma}' \rangle} = \{1\}$ and so an element α of the set $\overline{\langle \tilde{\gamma} \rangle} \cdot \overline{\langle \tilde{\gamma}' \rangle}$ admits a unique representation of the form $\alpha = \tilde{\gamma}^s \tilde{\gamma}'^{-t}$, with $s, t \in \mathbb{Z}_p$.

Lemma 2.8. *Let $\delta, \delta' \in \Pi_{g,n}$ be elements such that their free homotopy classes contain non-separating s.c.c.'s on the Riemann surface $S_{g,n}$. Let, respectively, $\overline{\langle \delta \rangle}$ and $\overline{\langle \delta' \rangle}$ be the closed subgroups which δ and δ' span in the pro- p completion $\Pi_{g,n}^{(p)}$. Then, it holds:*

$$\Pi_{g,n} \cap \overline{\langle \delta \rangle} \cdot \overline{\langle \delta' \rangle} = \langle \delta \rangle \cdot \langle \delta' \rangle.$$

Proof. Let us denote, for simplicity, the group $\Pi_{g,n}$ by Π and, for a group G , let us denote by G_k the k -th term of the lower central series. Then, the following facts are well known. The \mathbb{Z} -module $\mathcal{L}(\Pi) := \bigoplus_{i=1}^{\infty} \Pi_i / \Pi_{i+1}$ is free and has a natural structure of Lie algebra induced by the commutator bracket. Similarly, the \mathbb{Z}_p -module $\mathcal{L}(\Pi^{(p)}) := \prod_{i=1}^{\infty} \Pi_i^{(p)} / \Pi_{i+1}^{(p)}$ is free and has a natural structure of Lie algebra induced by the commutator bracket. The graded pieces of these two Lie algebras are related by natural isomorphisms, for all $i \geq 1$:

$$\Pi_i^{(p)} / \Pi_{i+1}^{(p)} \cong \Pi_i / \Pi_{i+1} \otimes \mathbb{Z}_p.$$

In particular, the elements of $\mathcal{L}(\Pi^{(p)})$ in the image of $\mathcal{L}(\Pi)$ are the elements of finite type whose graded components belong to the free \mathbb{Z} -submodules Π_i / Π_{i+1} , for all $i \geq 1$.

Let us fix for the fundamental group Π a standard ordered presentation:

$$\Pi = \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, u_1, \dots, u_n \mid \prod_{i=1}^g [\alpha_i, \beta_i] \cdot u_n \cdots u_1 \rangle,$$

where u_i , for $i = 1, \dots, n$, is a simple loop around the puncture P_i and $\alpha_1 = \delta$. In virtue of the Campbell-Hausdorff formula, this choice induces compatible bijective correspondences $\log: \Pi \xrightarrow{\sim} \mathcal{L}(\Pi)$ and $\log_p: \Pi^{(p)} \xrightarrow{\sim} \mathcal{L}(\Pi^{(p)})$, where the second map is also continuous.

Let us assume that it holds $\delta' \neq \delta^{\pm 1}$, otherwise the statement of the lemma is trivial. It holds $\log_p(\delta^s \delta'^t) = s \cdot \log_p(\delta) + \log_p(\delta'^t)$. Hence, the element $\log_p(\delta^s \delta'^t)$ is of finite type and with "discrete" homogeneous components only if t and then s belong to \mathbb{Z} . □

By Lemma 2.8, the pro-curves $\hat{\gamma}$ and $\hat{\gamma}'$ can intersect only in the image of $\mathbb{D} \times \Pi_{g,n}$ in the quotient $\mathbb{D} \times \Pi_{g,n}^{(p)} / \Pi_{g,n}$. This is the "discrete" leaf \mathbb{D} of the p -adic solenoid $\mathbb{S}_{g,n}^{(p)}$, where we already know that $\hat{\gamma}$ and $\hat{\gamma}'$ intersect only in the point P . This proves (*).

The identity (*) means that there is a normal subgroup L of $\Pi_{g,n}$ of index a power of p such that all intersection points of γ_L and γ'_L lie above the point $\overline{P} \in S_{g,n}$. So, they lie in the same orbit for the action of the deck transformation group G_L of the covering $\pi_L: S_L \rightarrow S_{g,n}$ and, for a fixed orientation on γ_L and γ'_L , the intersection indices are the same at all points of $\gamma_L \cap \gamma'_L$.

Therefore, if $\overline{\gamma}_L$ and $\overline{\gamma}'_L$ are cycles of the homology group $H_1(\overline{S}_L, \mathbb{Z})$ supported on γ_L and γ'_L , respectively, it holds $\langle \overline{\gamma}_L, \overline{\gamma}'_L \rangle_L = k$, for some integer $k \neq 0$. The same then holds for every finite index normal subgroup K of $\Pi_{g,n}$ contained in L . □

Let us now prove Theorem 2.3. By the remarks following Definition 2.1, it is easy to reduce the proof to the case when both γ and γ' are non-power. Let us assume as well, as usual, that both closed curves γ and γ' are geodesics for the given metric on $S_{g,n}$. In this way, the geometric intersection number $|\gamma \cap \gamma'|_G = 0$ if and only if either γ and γ' are disjoint, for instance when one of the two curves bounds a 1-punctured disc, or $\gamma = \gamma'$ is a s.c.c.. In all these cases, the conclusion of the theorem holds for any covering $S \rightarrow S_{g,n}$ and there is nothing to prove.

Let us then assume that $|\gamma \cap \gamma'|_G \neq 0$. By Lemma 2.5, there is a normal subgroup L of $\Pi_{g,n}$ of index a power of p such that each irreducible component of $\pi_L^{-1}(\gamma)$ and $\pi_L^{-1}(\gamma')$ is homotopic to a non-separating s.c.c. and then is a non-separating s.c.c.. Since it holds $|\gamma \cap \gamma'|_G \neq 0$, there are irreducible components of $\pi_L^{-1}(\gamma)$ and $\pi_L^{-1}(\gamma')$ which have non-trivial geometric intersection. By Lemma 2.7, there is then a characteristic subgroup K of L of index a power of p , with the property that, for every finite index subgroup K' of K , there are elements $x \in V_{K',\gamma}$ and $y \in V_{K',\gamma'}$ such that $\langle x, y \rangle_{K'} \neq 0$.

3 A group-theoretic characterization of simple closed curves

Given a standard presentation $\Pi_g = \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g \mid \prod_{i=1}^g [\alpha_i, \beta_i] \rangle$ of the fundamental group of a closed oriented surface S_g , there are various algorithms which permit to determine whether there is a s.c.c. in the free homotopy class of an element of Π_g (for a survey on this subject, see for instance §3 of [6]).

In this section, we show that this can be done group-theoretically, without the datum of a standard presentation of the group. We do so by combining the results of the previous section with some well known facts.

Let K be a finite index subgroup of Π_g . Then, the cup product on the homology group $H_1(K)$ can be recovered from the group structure of K . Indeed, it is easy to check that there is a short exact sequence of groups

$$0 \rightarrow \mathbb{Z} \xrightarrow{\phi} K/[K, K] \otimes K/[K, K] \xrightarrow{\psi} K/[[K, K]K] \rightarrow 1,$$

where the epimorphism ψ is induced by the assignment $\alpha \otimes \beta \mapsto \alpha\beta$, for all $\alpha, \beta \in K$, and the monomorphism ϕ is defined by $\phi(1) = \sum_{i=1}^{g_K} (a_i \otimes b_i - b_i \otimes a_i)$, with g_K the genus of S_K and $\{a_i, b_i\}$ is any standard set of generators for the homology group $H_1(S_K, \mathbb{Z})$.

The cap product on cohomology is then the dual of ϕ :

$$\phi^*: H^1(K) \otimes H^1(K) \rightarrow \mathbb{Z}.$$

Since ϕ^* is a non-degenerate skew-symmetric form, it establishes a canonical isomorphism between $H^1(K)$ and its dual which we can identify with $H_1(K)$. In this way, we get a description of the cup product on $H_1(K)$ purely in terms of the group structure of K .

By Corollary 2.4, the following principle holds:

Theorem 3.1. *Let S_g be a closed oriented surface of genus $g \geq 2$ and let γ be an element of its fundamental group $\pi_1(S_g, s_0)$. Then, the existence of an embedded representative in the free homotopy class of γ is determined by the group structure of $\pi_1(S_g, s_0)$.*

This can be further strengthened observing that, according to Corollary 2.4, to characterize s.c.c.'s on S_g , we just need to consider subgroups K of Π_g of index a power of p , for some fixed prime and that Corollary 2.4 holds true if, instead of homology with integral coefficients, we consider homology groups with any kind of non-torsion coefficients, for instance \mathbb{Z}_p coefficients. Moreover, by means of the Campbell-Hausdorff formula, the group structure of the pro- p completion $\Pi_g^{(p)}$ is determined by the p -adic Lie algebra $\mathcal{L}(\Pi_g^{(p)})$, introduced in the proof of Lemma 2.8. Hence, it holds as well:

Theorem 3.2. *Let S_g be a closed oriented surface of genus $g \geq 2$ and let γ be an element of its fundamental group Π_g . Then, the existence of an embedded representative in the free homotopy class of γ is determined by the Lie algebra structure of $\mathcal{L}(\Pi_g^{(p)})$.*

Theorem 3.2 points in the direction of the characterization of s.c.c.'s given in the papers [2], [3], [4], [5].

4 Distinguishing closed curves

As an elementary consequence of Theorem 2.3, we derive the following result:

Theorem 4.1. *Let γ and γ' be powers of s.c.c.'s on the Riemann surface $S_{g,n}$, such that not both of them bound a 1-punctured disc. Then, the two curves are non-homotopic if and only if, for any fixed prime p , there is a normal, unramified p -covering $\pi_K: S_K \rightarrow S_{g,n}$ such that the G_K -invariant submodules $V_{K,\gamma}$ and $V_{K,\gamma'}$ of $H_1(\overline{S}_K)$ are distinct.*

Proof. One implication is obvious. So let us suppose that, for all normal, finite, unramified p -coverings $\pi_K: S_K \rightarrow S_{g,n}$, it holds $V_{K,\gamma} = V_{K,\gamma'}$. In particular, it holds $\langle x, y \rangle_K = 0$, for all $x \in V_{K,\gamma}$, $y \in V_{K,\gamma'}$ and all normal subgroups K of $\Pi_{g,n}$ of index a power of p .

From Theorem 2.3, it follows that γ and γ' have trivial geometric intersection. But then, for powers of s.c.c.'s not bounding a 1-punctured disc, with trivial geometric intersection, the above condition implies that a power of γ is homotopic to a power of γ' .

If a power γ^s of γ is homotopic to a power γ'^t of γ' and γ is homologically non-trivial on the closed surface $S_g = \overline{S}_{g,n}$, the homology classes of γ and γ' generate the same submodule of $H_1(S_g)$ only if $s = t$ and the curves γ and γ' are homotopic. If instead γ is homologically trivial on the closed surface S_g , after possibly iterating two abelian, unramified, p -coverings, we get a covering $\pi_K: S_K \rightarrow S_{g,n}$ such that all irreducible components of $\pi_K^{-1}(\gamma)$ are mapped bijectively on γ by π_K and have non-trivial integral homology classes in $H_1(\overline{S}_K)$. Then, $V_{K,\gamma} = V_{K,\gamma'}$ implies that $s = t$ and that γ and γ' are homotopic. □

For arbitrary closed curves, there is the following weaker separation property:

Theorem 4.2. *Let α and β be closed curves on $S_{g,n}$, not both of them bounding a 1-punctured disc. Then, the two curves are non-homotopic if and only if, for any fixed prime p , there is a normal, unramified p -covering $\pi_K: S_K \rightarrow S_{g,n}$ such that a cycle of $H_1(\overline{S}_K)$ supported on an irreducible component of $\pi_K^{-1}(\alpha)$ is distinct from all those supported on an irreducible component of $\pi_K^{-1}(\beta)$.*

Proof. The case when a power of one curve is homotopic to a power of the other can be treated as in the proof of Theorem 4.1. Let us then assume that no power of one curve is homotopic to a power of the other.

By Lemma 2.5, there is a finite index normal subgroup L of $\Pi_{g,n}$ of index a power of p such that each irreducible component of $\pi_L^{-1}(\alpha)$ and $\pi_L^{-1}(\beta)$ is homotopic to a power of a simple closed curve. Since no power of α is homotopic to a power of β , all irreducible components of $\pi_L^{-1}(\alpha)$ and $\pi_L^{-1}(\beta)$ are two by two non-homotopic.

At this point, we can apply Theorem 4.1 and conclude that there is a characteristic, finite, unramified p -covering $\pi_{K,L}: S_K \rightarrow S_L$ such that, if γ and γ' denote irreducible components of $\pi_L^{-1}(\alpha)$ and $\pi_L^{-1}(\beta)$, respectively, the K/L -invariant submodules V_γ and $V_{\gamma'}$ of $H_1(\overline{S}_K)$, generated by the cycles supported on the irreducible components of $\pi_{K,L}^{-1}(\gamma)$ and $\pi_{K,L}^{-1}(\gamma')$, are distinct. In particular, this implies the conclusion of the theorem. \square

Let p be a prime. A group G is conjugacy p -separable if, whenever x and y are non-conjugate elements of G , there exists some finite p -quotient of G in which the images of x and y are non-conjugate.

An almost immediate consequence of Theorem 4.2 is conjugacy p -separability of fundamental groups of oriented Riemann surface. This was well known for open surfaces but, in the closed surface case, it was proved only recently by Paris [9].

Theorem 4.3. *The fundamental group $\Pi_{g,n}$ of an oriented surface is conjugacy p -separable.*

Proof. The cases in which $\Pi_{g,n}$ is abelian are trivial. So, let us assume $2g - 2 + n > 0$. Let then $\alpha, \beta \in \Pi_{g,n}$ belong to distinct conjugacy classes. In case α and β are freely homotopic to s.c.c.'s bounding 1-punctured discs, let N be the kernel of the natural epimorphism $\Pi_{g,n} \rightarrow \Pi_g$ induced by filling in the punctures of $S_{g,n}$ and let $L := N[\Pi_{g,n}, \Pi_{g,n}]\Pi_{g,n}^p$.

The images of α and β in the homology group $H_1(L, \mathbb{Z}/p)$ are in distinct \tilde{G}_L -orbits. Hence, their images in the finite p -quotient $\Pi_{g,n}/[L, L]L^p$, which are contained in the subgroup $L/[L, L]L^p \cong H_1(L, \mathbb{Z}/p)$, are non-conjugate.

Let us then assume that the situation above does not occur. By Theorem 4.1, there is a characteristic subgroup K of $\Pi_{g,n}$ of index a power of p and a positive integer m such that the G_K -orbits of $\tilde{\alpha}$ and $\tilde{\beta}$ in $H_1(K, \mathbb{Z}/p^m)$ are distinct, where $\tilde{\alpha}$ and $\tilde{\beta}$ denote the images in $H_1(K, \mathbb{Z}/p^m)$ of the minimal positive powers α^s and β^t contained in K . We can assume that $s = t$, otherwise it is already clear that the images of α and β in the p -quotient $\Pi_{g,n}/K$ are not in the same conjugacy class.

The subgroup $[K, K]K^{p^m}$ of K generated by commutators and p^m powers is a characteristic subgroup of $\Pi_{g,n}$ of index a power of p , such that the image of K in the p -group

$\Pi_{g,n}/[K, K]K^{p^m}$ is naturally isomorphic to the homology group $H_1(K, \mathbb{Z}/p^m)$. Therefore, the images $\tilde{\alpha}$ and $\tilde{\beta}$ of α^s and β^s in this p -group belong to distinct conjugacy classes. The same then is true for the images there of α and β .

□

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